

# Generation of modules and transcendence degree of zero-cycles

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## Abstract

We construct an example of a regular algebra over  $\mathbb{C}$  of dimension  $d$  and a rank  $r$  projective module over it which is not generated by  $d + r - 1$  elements. This strengthens an example by Swan over the field of real numbers.

*To I. R. Shafarevich, with a profound respect*

## 1 Introduction

Let  $R$  be a unital finitely generated commutative algebra over a field  $k$  and let  $M$  be a finitely generated projective  $R$ -module of rank  $r$ . Let  $d$  be the Krull dimension of  $R$ . It follows from the Forster–Swan theorem [2], [6] that  $M$  is generated by  $d + r$  elements over  $R$  (actually, the theorem deals with rings and modules of a more general type). In our particular case this result can be also easily deduced from the Bertini theorem.

A natural question is whether the lower bound  $d + r$  on the number of generators is exact. In other words, whether there exist  $R$  and  $M$  as above such that  $M$  is not generated by  $d + r - 1$  elements (the question is trivial for the case  $d = 1$ ). Swan [5, Th.4] has constructed such examples for all  $d, r$  and  $k = \mathbb{R}$ . His argument essentially uses that the field  $\mathbb{R}$  is not algebraically closed. Up to the author's knowledge, there was no an example in the literature with an algebraically closed field  $k$ . The aim of this paper is to construct examples of this type.

Using Chern classes with values in Chow groups, we reduce the initial problem to a question about a non-vanishing of elements in Chow groups of zero-cycles on affine varieties (this idea closely follows [5], where Stiefel–Whitney classes are used). Then we apply transcendence degree of zero-cycles [3] in order to get the non-vanishing. This approach is essentially based on the Bloch–Srinivas decomposition of diagonal [1].

Examples of algebras and modules as in this paper can be further used in order to construct new non-trivial examples of Jordan algebras with the help of a method from [7].

However, this requires that  $R$  and  $M$  satisfy certain additional conditions related to derivations. The actual construction of such examples remains an open problem.

This note arose from discussions with C. Shramov of a related question posed by V. N. Zhelyabin. The author is deeply grateful to both of them for a stimulating and friendly atmosphere during their common stay in Novosibirsk in Fall 2011. The author was partially supported by RFBR grant 11-01-00145-a, MK-4881.2011.1, NSh grant 5139.2012.1, and by AG Laboratory HSE, RF gov. grant, ag. 11.G34.31.0023.

## 2 Main result and reduction to zero-cycles

Here is the main result of the paper.

**Theorem 1.** *For all natural numbers  $d$  and  $r$ , there exist a regular finitely generated algebra  $R$  over  $\mathbb{C}$  of dimension  $d$  and a projective finitely generated  $R$ -module  $M$  of rank  $r$  such that  $M$  is not generated over  $R$  by  $d + r - 1$  elements.*

Let  $k$  be again an arbitrary field, put  $U := \text{Spec}(R)$ , let  $N$  be a rank one projective  $R$ -module,  $M = N \oplus R^{\oplus(r-1)}$ , and let  $L$  be the line bundle on  $U$  that corresponds to  $N$ . For an algebraic variety  $X$  over  $k$ , by  $CH^p(X)$  denote the Chow group of codimension  $p$  cycles on  $X$ . Given a vector bundle  $E$  on  $X$ , by  $c_p(E) \in CH^p(X)$  denote the corresponding Chern class. The following lemma is a direct analogue of an argument from [5, Ex.2].

**Lemma 2.** *Suppose that  $M$  is generated by  $d + r - 1$  elements. Then we have that  $c_1(L)^d = 0$  in  $CH^d(U)$ .*

*Proof.* By the condition of the lemma, there is an exact triple of vector bundles on  $U$

$$0 \rightarrow E \rightarrow \mathcal{O}_U^{\oplus(d+r-1)} \rightarrow L \oplus \mathcal{O}_U^{\oplus(r-1)} \rightarrow 0.$$

By the Whitney formula, we have

$$1 + c_1(E) + \dots + c_d(E) = (1 + c_1(L))^{-1}.$$

In particular,  $c_d(E) = (-1)^d c_1(L)^d$ . On the other hand, since the rank of  $E$  equals  $d - 1$ , we obtain  $c_d(E) = 0$ .  $\square$

Thus in order to construct our main example it would be helpful to show a non-vanishing of elements in  $CH^d(U)$ . It is important that the Chern classes here are with values in Chow groups and not in cohomology groups (Betti, de Rham, or étale), because degree  $2d$  cohomology groups are trivial for any affine variety of dimension  $d$ .

The following is an interpretation of the example [5, Ex.2, Th.4] in terms of Chow groups.

*Example 3.* Let  $Q \subset \mathbb{P}^d$  be a smooth projective quadric without rational points over  $k$ ,  $U = \text{Spec}(R)$  be the complement to  $Q$  in  $\mathbb{P}^d$ ,  $L = \mathcal{O}_{\mathbb{P}^d}(1)|_U$ ,  $N$  be the corresponding  $R$ -module, and let  $M = N \oplus R^{\oplus(r-1)}$ . Then  $c_1(L)^d$  is the restriction to  $U$  of the class of a point in  $CH^d(\mathbb{P}^d) \cong \mathbb{Z}$ . If  $c_1(L)^d = 0$ , then it follows from the exact sequence

$$CH^{d-1}(Q) \rightarrow CH^d(\mathbb{P}^d) \rightarrow CH^d(U) \rightarrow 0$$

that the quadric  $Q$  has a degree one zero-cycle over  $k$ . By a well-known result of Springer [4, Ch.VII, Th.2.3], this contradicts with the absence of  $k$ -points on  $Q$  (for  $k = \mathbb{R}$ , we also obtain this contradiction considering the action of the complex conjugation on an effective zero-cycle of odd degree). Therefore by Lemma 2,  $M$  is not generated by  $d + r - 1$  elements.

### 3 Transcendence of zero-cycles and the proof of the main result

Let us recall some notions and facts from [3]. Given a subfield  $k_0 \subset k$  and a variety  $X_0$  over  $k_0$ , by  $(X_0)_k$  denote the extension of scalars of  $X_0$  from  $k_0$  to  $k$ :

$$(X_0)_k = \text{Spec}(k) \times_{\text{Spec}(k_0)} X_0.$$

Let  $X$  be a variety over  $k$ . For simplicity, assume that  $X$  is irreducible. Let  $d$  be the dimension of  $X$ . Suppose that  $X$  is the extension of scalars of a variety  $X_0$  defined over a subfield  $k_0 \subset k$ , i.e., it is fixed an isomorphism  $X \cong (X_0)_k$ . We say that an open (respectively, closed) subset in  $X$  is defined over  $k_0$  if it is the extension of scalars of an open (respectively, closed) subset in  $X_0$ .

A point  $P \in X(k) = X_0(k)$  corresponds to a morphism of schemes  $P: \text{Spec}(k) \rightarrow X_0$ . By  $\xi_P$  denote its image and put

$$\text{tr.deg}(P/k_0) := \text{tr.deg}(k_0(\xi_P)/k_0).$$

Explicitly,  $\text{tr.deg}(P/k_0)$  is the transcendence degree over  $k_0$  of the field generated by coordinates of  $P$  with respect to an arbitrary affine open neighborhood defined over  $k_0$ . Given an element  $\alpha \in CH^d(X)$ , its transcendence degree  $\text{tr.deg}(\alpha/k_0)$  is the minimal natural number  $n$  such that there exists a zero-cycle  $\sum_i n_i P_i$  on  $X$  over  $k$  that represents  $\alpha$  and such that for any  $i$ , we have  $\text{tr.deg}(P_i/k_0) \leq n$ . One defines similarly transcendence degree for elements in  $CH^d(X)_{\mathbb{Q}} := CH^d(X) \otimes_{\mathbb{Z}} \mathbb{Q}$  (one considers all representatives with rational coefficients).

*Remark 4.* In the above notation, assume that there is an open subset  $U \subset X$  defined over  $k_0$  such that  $\alpha|_U = 0$  in  $CH^d(U)$ . Then  $\alpha$  is represented by a zero-cycle supported on  $Z = X \setminus U$ . Since the dimension of  $Z$  is strictly less than  $d$  and  $Z$  is defined over  $k_0$ , it follows from the definition of a transcendence degree of zero-cycles that  $\text{tr.deg}(\alpha/k_0) < d$ . This remains true when Chow groups are taken with rational coefficients.

The next statement follows directly from [3, Th.7].

**Proposition 5.** *Let  $X$  be an irreducible smooth projective variety of dimension  $d$  over a field of characteristic zero  $k$ . Assume that  $X$  is the extension of scalars of a variety  $X_0$  defined over a subfield  $k_0 \subset k$ . Assume that any finitely generated field over  $k_0$  can be embedded into  $k$  over  $k_0$  (equivalently,  $k$  is algebraically closed and of infinite transcendence degree over  $k_0$ ). Suppose that there is a point  $P \in X(k)$  such that  $\text{tr.deg}(P/k_0) = d$  and the class  $[P]$  of  $P$  in  $CH^d(X)_{\mathbb{Q}}$  satisfies  $\text{tr.deg}([P]/k_0) < d$ . Then  $H^0(X, \Omega_X^d) = 0$ .*

The idea of the proof of Proposition 5 is to consider  $[P]$  as the restriction of the class of the diagonal  $X_0 \times X_0$  to the subscheme  $X_0 \times \text{Spec}(k_0(X_0))$  and to use the Bloch–Srinivas decomposition of diagonal [1].

Now we are ready to prove Theorem 1. Let  $C$  be a smooth projective curve over  $\bar{\mathbb{Q}}$  of positive genus and put  $X_0 = C^d$ ,  $X = (X_0)_{\mathbb{C}}$ . Consider a complex point  $P = (P_1, \dots, P_d) \in X(\mathbb{C})$  that does not belong to any subvariety in  $X$  defined over  $\bar{\mathbb{Q}}$  except for  $X$  itself. Such a point exists, because there are countably many subvarieties in  $X$  defined over  $\bar{\mathbb{Q}}$ , while  $\mathbb{C}$  is uncountable. Explicitly,  $P$  satisfies the following condition: choose an affine open subset in  $C$  defined over  $\bar{\mathbb{Q}}$  whose extension of scalars to  $\mathbb{C}$  contains  $P_1, \dots, P_d$ . One requires that the coordinates of  $P_1, \dots, P_d$  with respect to this affine chart generate a field of transcendence degree  $d$  over  $\bar{\mathbb{Q}}$ .

Further, consider the following divisors in  $X$ :

$$D_i := \{(x_1, \dots, x_d) \in X \mid x_i = P_i\}, \quad 1 \leq i \leq d$$

and a (reducible) divisor  $D = \cup_i D_i$ . Note that  $D$  is not defined over  $\bar{\mathbb{Q}}$ . Finally, let  $U = \text{Spec}(R)$  be any non-empty affine open subset in  $X$  defined over  $\bar{\mathbb{Q}}$ ,  $L = \mathcal{O}_X(D)|_U$ ,  $N$  be the corresponding  $R$ -module, and let  $M = N \oplus R^{\oplus(r-1)}$ . Then we have

$$c_1(L)^d = c_1(\mathcal{O}_X(D))^d|_U = d! \cdot [P]|_U \in CH^d(U).$$

Suppose that  $[P]|_U = 0$  in  $CH^d(U)_{\mathbb{Q}}$ . By Remark 4, we obtain  $\text{tr.deg}([P]/k_0) < d$  in  $CH^d(X)_{\mathbb{Q}}$ . By Proposition 5, this contradicts with the condition

$$H^0(X, \Omega_X^d) \cong H^0(C, \Omega_C^1)^{\otimes d} \neq 0.$$

Therefore,  $[P]|_U \neq 0$  in  $CH^d(U)_{\mathbb{Q}}$ . Consequently,  $d! \cdot [P]|_U \neq 0$  in  $CH^d(U)$  and by Lemma 2,  $M$  is not generated by  $d + r - 1$  elements over  $R$ . This finishes the proof of Theorem 1.

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